

The Method of Multiple Combinatorial Telescoping

Daniel K. Du¹, Qing-Hu Hou^{1,2} and Charles B. Mei²

¹Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

²Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

E-mail: daniel@tju.edu.cn, hou@nankai.edu.cn, meis@mail.nankai.edu.cn

Abstract

We generalize the method of combinatorial telescoping to the case of multiple summations. We shall demonstrate this idea by giving combinatorial proofs for two identities of Andrews on parity indices of partitions.

AMS Classification: 05A17, 11P83

Keywords: Integer partitions, parity index of partition, combinatorial telescoping, multiple combinatorial telescoping.

1 Introduction

The method of combinatorial telescoping for alternating sums was proposed by Chen et al. [3], which can be used to show that an alternating sum satisfies certain recurrence relation combinatorially. With this method, one give combinatorial interpretations for many q -series identities such as Watson's identity [10] and Sylvester's identity [9]. For q -series identities on positive terms, Chen et al. [6] presented the corresponding combinatorial telescoping, based on which they established a combinatorial proof for an identity due to Andrews [2].

In this paper, we shall generalize the method of combinatorial telescoping to the multiple cases. More precisely, we shall give the combinatorial telescoping for q -series identities of the following form

$$\sum_{\mathbf{k}=0}^{\infty} (-1)^{\delta \cdot \mathbf{k}} f(\mathbf{k}) = \sum_{\mathbf{k}=0}^{\infty} (-1)^{\delta \cdot \mathbf{k}} g(\mathbf{k}), \quad (1.1)$$

where $\mathbf{k} = (k_1, \dots, k_m)$ and $\delta = (\delta_1, \dots, \delta_m)$ are m -dimensional vectors, and $\delta_i \in \{0, 1\}$.

Assume that $f(\mathbf{k})$ and $g(\mathbf{k})$ are weighted counts of sets $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$, respectively, that is,

$$f(\mathbf{k}) = \sum_{\alpha \in A_{\mathbf{k}}} w(\alpha) \quad \text{and} \quad g(\mathbf{k}) = \sum_{\alpha \in B_{\mathbf{k}}} w(\alpha).$$

Motivated by the idea of creative telescoping of Zeilberger [7, 8, 12], we will construct sets $\{H_{i,\mathbf{k}}\}_{i=1}^m$ with a weight assignment w such that there exists a sequence of weight preserving bijections

$$\phi_{\mathbf{k}} : A_{\mathbf{k}} \bigcup_{\{i|\delta_i=0\}} H_{i,\mathbf{S}_i\mathbf{k}} \longrightarrow B_{\mathbf{k}} \bigcup_{i=1}^m H_{i,\mathbf{k}} \bigcup_{\{i|\delta_i=1\}} H_{i,\mathbf{S}_i\mathbf{k}}, \quad (1.2)$$

where \mathbf{S}_i is the shift operator on the i -th part, i.e.,

$$\mathbf{S}_i \mathbf{k} = (k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_m).$$

Since $\phi_{\mathbf{k}}$ and $\phi_{\mathbf{S}_i \mathbf{k}}$ are weight preserving for $1 \leq i \leq m$, both

$$\phi_{\mathbf{k}}(H_{i, \mathbf{S}_i \mathbf{k}}) \quad \text{and} \quad \phi_{\mathbf{S}_i \mathbf{k}}(H_{i, \mathbf{S}_i \mathbf{k}})$$

have the same weight as $H_{i, \mathbf{S}_i \mathbf{k}}$. Realizing that for all $1 \leq i \leq m$ with $\delta_i = 1$, we have

$$\phi_{\mathbf{k}}^{-1}(H_{i, \mathbf{S}_i \mathbf{k}}) \subseteq A_{\mathbf{k}} \bigcup_{\{i | \delta_i = 0\}} H_{i, \mathbf{S}_i \mathbf{k}}$$

and

$$\phi_{\mathbf{S}_i \mathbf{k}}^{-1}(H_{i, \mathbf{S}_i \mathbf{k}}) \subseteq A_{\mathbf{S}_i \mathbf{k}} \bigcup_{\{i | \delta_i = 0\}} H_{i, \mathbf{S}_i^2 \mathbf{k}},$$

which implies that the corresponding summands will cancel each other in the desired identity (1.1). More precisely, if we set

$$h_i(\mathbf{k}) = \sum_{\alpha \in H_{i, \mathbf{k}}} w(\alpha), \quad (1 \leq i \leq m),$$

then the bijection (1.2) implies that

$$\begin{aligned} & f(\mathbf{k}) + \sum_{\{i | \delta_i = 0\}} h_i(\mathbf{S}_i \mathbf{k}) \\ &= g(\mathbf{k}) + \sum_{i=1}^m h_i(\mathbf{k}) + \sum_{\{i | \delta_i = 1\}} h_i(\mathbf{S}_i \mathbf{k}) \\ &= g(\mathbf{k}) + \sum_{\{i | \delta_i = 0\}} h_i(\mathbf{k}) + \sum_{\{i | \delta_i = 1\}} h_i(\mathbf{k}) + \sum_{\{i | \delta_i = 1\}} h_i(\mathbf{S}_i \mathbf{k}). \end{aligned} \quad (1.3)$$

We assume, like the conditions for the creative telescoping [7, 8, 12], that $H_i(\mathbf{0}) = \emptyset$ and $H_i(\mathbf{k})$ vanishes for sufficiently large \mathbf{k} for $1 \leq i \leq m$. Multiplying $(-1)^{\delta \cdot \mathbf{k}}$ and summing over \mathbf{k} on both sides of (1.3), since

$$(-1)^{\delta \cdot \mathbf{k}} h_i(\mathbf{S}_i \mathbf{k}) + (-1)^{\delta \cdot \mathbf{S}_i \mathbf{k}} h_i(\mathbf{S}_i \mathbf{k}) = 0$$

for $1 \leq i \leq m$ with $\delta_i = 1$, we will obtain the identity (1.1), which is often an identity we wish to establish.

Indeed, once we have bijections $\phi_{\mathbf{k}}$ in (1.2), combining all these bijections, we are lead to a correspondence

$$\phi : A \cup C \rightarrow B \cup C, \quad (1.4)$$

where

$$A = \bigcup_{\mathbf{k}=\mathbf{0}}^{\infty} A_{\mathbf{k}}, \quad B = \bigcup_{\mathbf{k}=\mathbf{0}}^{\infty} B_{\mathbf{k}}, \quad \text{and} \quad C = \bigcup_{\{i | \delta_i = 0\}} H_{i, \mathbf{S}_i \mathbf{k}}.$$

To be more specific, we can derive a bijection

$$\phi : A \cup C \setminus D \longrightarrow B \cup C,$$

where

$$D = \bigcup_{\mathbf{k}=\mathbf{0}}^{\infty} \bigcup_{\{i | \delta_i = 1\}} \phi_{\mathbf{k}}^{-1}((H_{i, \mathbf{k}} \cup H_{i, \mathbf{S}_i \mathbf{k}})),$$

and an involution

$$\psi : D \longrightarrow D$$

given by $\phi(\alpha) = \phi_{\mathbf{k}}(\alpha)$ if $\alpha \in A_{\mathbf{k}} \cup C$ and

$$\psi(\alpha) = \begin{cases} \phi_{\mathbf{S}_i^{-1}\mathbf{k}}^{-1} \phi_{\mathbf{k}}(\alpha), & \text{if } \alpha \in \phi^{-1}(H_{i,\mathbf{k}}), \\ \phi_{\mathbf{S}_i\mathbf{k}}^{-1} \phi_{\mathbf{k}}(\alpha), & \text{if } \alpha \in \phi^{-1}(H_{i,\mathbf{S}_i\mathbf{k}}). \end{cases}$$

As shown in [6], using the method of cancelation (see [5]), the bijection ϕ in (1.4) implies a bijection

$$\psi : A \rightarrow B,$$

which gives a combinatorial interpretation of the desired q -series identity (1.1).

The above approach to proving an identity like (1.1) is called *multiple combinatorial telescoping*. To illustrate the idea of this method, we shall prove two q -series identities proposed by Andrews [2].

In the study of parity in partition identities, Andrews [2] proposed fifteen problems. Two of them, labeled as Question 9 and Question 10 in [2], asked for proving the following two identities of sum on double variables:

$$\sum_{n,k \geq 0} \frac{(-1)^n q^{(n-k)^2 + k^2 + n - k}}{(-q; q)_n (q; q)_{2k-1} (q; q)_{n-2k+1}} = \sum_{n=1}^{\infty} (-1)^n q^{n^2}, \quad (1.5)$$

$$\sum_{n,k \geq 0} \frac{(-1)^n q^{(n-k)^2 + k^2 + n + k}}{(-q; q)_n (q; q)_{2k} (q; q)_{n-2k}} = \sum_{n=0}^{\infty} (-1)^n q^{n^2}. \quad (1.6)$$

Recently Yee [11] and Chu [4] provided algebraic proofs for (1.5) and (1.6) independently, while the combinatorial interpretation is still open.

In the framework of the method of multiple combinatorial telescoping, we shall give a more extensive result as follows.

Theorem 1.1

$$\sum_{m,k \geq 0} \frac{(-a)^m q^{(m-k)^2 + k^2 + m - k}}{(aq^2; q^2)_m} \left[\begin{matrix} m \\ 2k-1 \end{matrix} \right]_q = \sum_{n \geq 1} (-a)^n q^{n^2}, \quad (1.7)$$

$$\sum_{m,k \geq 0} \frac{(-a)^m q^{(m-k)^2 + k^2 + k}}{(aq^2; q^2)_m} \left[\begin{matrix} m \\ 2k \end{matrix} \right]_q = \sum_{n \geq 0} (-a)^n q^{n^2}, \quad (1.8)$$

By setting $a = 1$ in (1.7) and (1.8), they reduce to (1.5) and (1.6), respectively, which give combinatorial answers to the open questions of Andrews.

2 Multiple Combinatorial Telescoping for Identities of Andrews

In this section, by constructing multiple combinatorial telescoping, we shall give certain bijections for recurrence relations of identity (1.7) and (1.8), respectively. Based on the method of cancelation (see [5]), these bijections lead to corresponding combinatorial interpretations of Question 9 and Question 10

proposed by Andrews in [2]. Since the proofs of (1.7) and (1.8) are similar, we introduce the procedure for (1.7) and a sketch proof for (1.8).

Let us recall some definitions concerning partitions as used in Andrews [1]. A partition is a non-increasing finite sequence of positive integers

$$\lambda = (\lambda_1, \dots, \lambda_l).$$

The integers λ_i are called the parts of λ . The sum of parts and the number of parts are denoted by

$$|\lambda| = \lambda_1 + \dots + \lambda_l,$$

and $\ell(\lambda) = l$, respectively. The number of k -parts in λ is denoted by $m_k(\lambda)$. The special partition with no parts is denoted by \emptyset . We shall use diagrams to represent partitions and use rows to represent parts.

In order to prove identity (1.7), we let

$$F(q) = \sum_{m,k} \frac{(-a)^m q^{(m-k)^2 + k^2 + m - k}}{(aq^2; q^2)_m} \left[\begin{matrix} m \\ 2k-1 \end{matrix} \right]_q,$$

and the corresponding summand

$$F_{m,k} = \frac{(-a)^m q^{(m-k)^2 + k^2 + m - k}}{(aq^2; q^2)_m} \left[\begin{matrix} m \\ 2k-1 \end{matrix} \right]_q. \quad (2.1)$$

Define $P_{m,k}$ to be the set of triples (τ, λ, μ) , where τ is a partition with only one part $(m-k)^2 + k^2 + m - k$, λ is a partition with no more than $2k-1$ parts and each part not exceeding $m-2k+1$, and μ is a partition with only even not exceeding $2m$; see Figure 1. In particular, when $m = k = 0$, we have $P_{0,0} = \emptyset$.

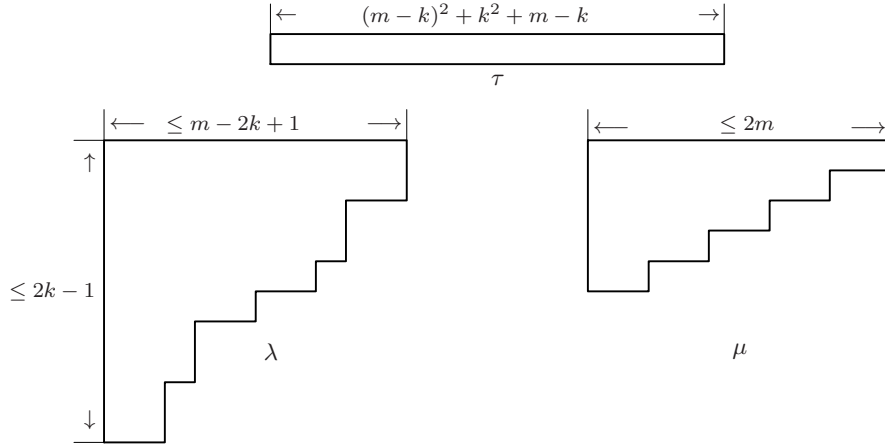


Figure 1: Illustration of an element $(\tau, \lambda, \mu) \in P_{m,k}$.

Moreover, one can see that the (m, k) -th summand $F_{m,k}$ of $F(q)$ in (2.1) can be viewed as the weight of $P_{m,k}$, that is,

$$\sum_{(\tau, \lambda, \mu) \in P_{m,k}} a^{m + \ell(\mu)} q^{|\tau| + |\lambda| + |\mu|}.$$

According to the exponent of a in the above definition, we divide $P_{m,k}$ into a disjoint union of subsets

$$P_{n,m,k} = \{(\tau, \lambda, \mu) \in P_{m,k} : \ell(\mu) = n - m\},$$

with $P_{n,m,k} = \emptyset$ when $m = k = 0$ or $2k - 1 > m > n$. By constructing three sets of triples (τ, λ, μ) as follows

$$\begin{aligned} G_{n,m,k} &= \{(\tau, \lambda, \mu) \in P_{n,m,k} : \ell(\lambda) = 2k - 1, m_2(\mu) = 0\}, \\ H_{n,m,k} &= \{(\tau, \lambda, \mu) \in P_{n,m,k} : \ell(\lambda) = 2k - 2, m_2(\mu) = 0\}, \\ K_{n,m,k} &= \{(\tau, \lambda, \mu) \in P_{n,m,k} : \ell(\lambda) \leq 2k - 3, m_2(\mu) = 0\}, \end{aligned}$$

we have the following combinatorial telescoping relation for $P_{n,m,k}$.

Theorem 2.1 *For any positive integer n and nonnegative integers m and k , there is a bijection*

$$\begin{aligned} \phi_{n,m,k} : \\ P_{n,m,k} \cup \{2n - 1\} \times P_{n-1,m,k} \cup K_{n,m+1,k} \rightarrow \\ G_{n,m,k} \cup G_{n,m+1,k} \cup H_{n,m,k} \cup H_{n,m+1,k} \cup K_{n,m,k} \cup K_{n,m+1,k+1} \cup K_{n,m+1,k}. \end{aligned}$$

Proof. Let

$$U_{n,m,k} = \{(\tau, \lambda, \mu) \in P_{n,m,k} : \ell(\lambda) \leq 2k - 2, m_2(\mu) \neq 0\},$$

and

$$T_{n,m,k} = \{(\tau, \lambda, \mu) \in P_{n,m,k} : \ell(\lambda) = 2k - 1, m_2(\mu) \neq 0\},$$

are two sets of triples (τ, λ, μ) , then we can divide $P_{n,m,k}$ into a disjoint union of five subsets, that is

$$P_{n,m,k} = G_{n,m,k} \cup H_{n,m,k} \cup K_{n,m,k} \cup U_{n,m,k} \cup T_{n,m,k}.$$

Now we construct bijections $\phi_{n,m,k}$ like (1.2). To this purpose, we can classify

$$P_{n,m,k} \cup \{2n - 1\} \times P_{n-1,m,k} \cup K_{n,m+1,k}$$

into four cases as below. We shall show that the elements in the first case are fix points, while the bijections for other three cases as follows:

$$\begin{aligned} \varphi_1 : \quad \{2n - 1\} \times F_{n-1,m,k} &\rightarrow G_{n,m+1,k}, \\ \varphi_2 : \quad U_{n,m,k} &\rightarrow H_{n,m+1,k}, \\ \varphi_3 : \quad T_{n,m,k} &\rightarrow K_{n,m+1,k+1}. \end{aligned}$$

To be more specific, we have the following four cases.

Case 0. For $(\tau, \lambda, \mu) \in G_{n,m,k} \cup H_{n,m,k} \cup K_{n,m,k} \cup K_{n,m+1,k}$, let the image be itself.

Case 1. For $(\tau, \lambda, \mu) \in P_{n-1,m,k}$, as $\ell(\lambda) = 2k - 1$, we add a column equals $2k - 1$ to λ and obtain a new partition λ' , where $\ell(\lambda') = 2k - 1$ and $\lambda'_1 \leq m - 2k + 2$. By adding $2m - 2k + 2$ to τ we get $\tau' = ((m - k + 1)^2 + k^2 + m - k + 1)$. And we can add 2 columns equal $n - m - 1$ to μ without changing the number of parts, thus the new partition μ' satisfies that $\ell(\mu') = n - m - 1$, $\mu'_1 \leq 2m + 2$ and $m_2(\mu') = 0$. We see that the weight of (τ, λ, μ) is less than (τ', λ', μ') by $2n - 1$. So we obtain the bijection $\varphi_1 : \{2n - 1\} \times P_{n-1,m,k} \rightarrow G_{n,m+1,k}$ defined by $(2n - 1, (\tau, \lambda, \mu)) \mapsto (\tau', \lambda', \mu')$. Figure 2 gives an illustration of the correspondence φ_1 .

Case 2. For any $(\tau, \lambda, \mu) \in U_{n,m,k}$, we add a column equals $2k - 2$ to λ to obtain a new partition λ' , where $\ell(\lambda') = 2k - 2$ and $\lambda'_1 \leq m - 2k + 2$. By adding

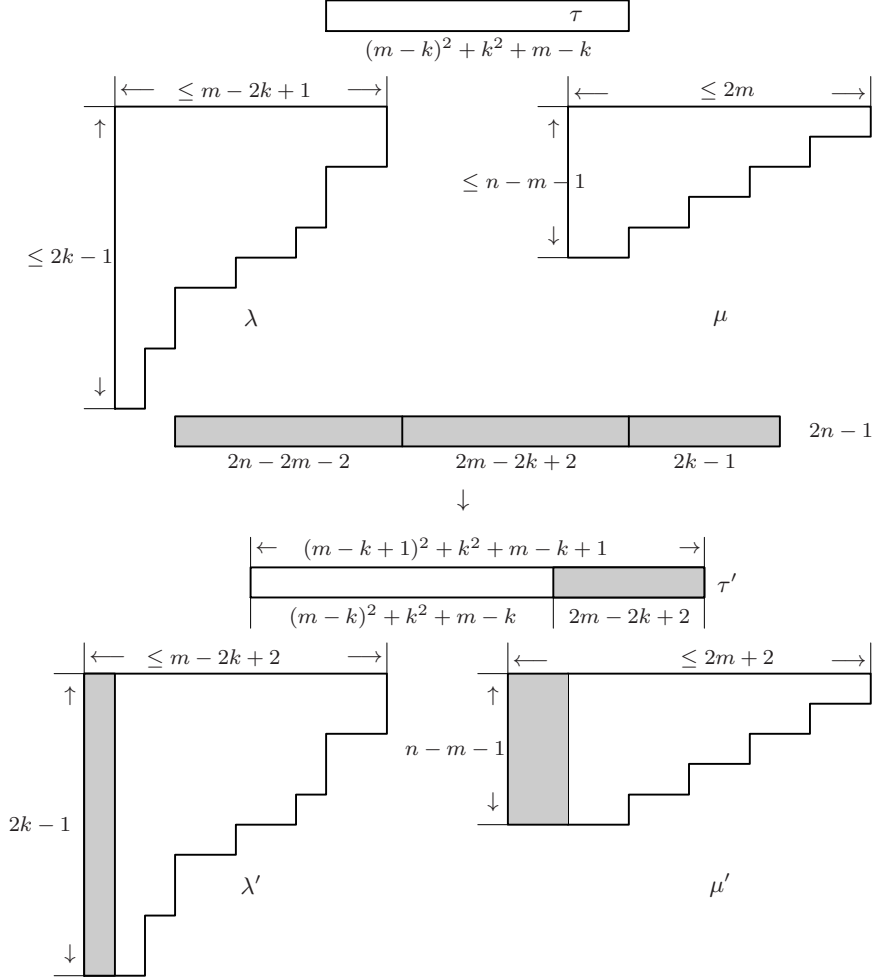


Figure 2: The bijection φ_1 in Case 1.

$2m - 2k + 2$ to τ we get $\tau' = ((m - k + 1)^2 + k^2 + m - k + 1)$. And we remove a $2m$ part from μ and getting μ' whose largest part is less than or equal to $2m$ and length equals to $n - m - 1$. This leads to the bijection $\varphi_2 : U_{n,m,k} \rightarrow H_{n,m+1,k}$ as given by $(\tau, \lambda, \mu) \mapsto (\tau', \lambda', \mu')$. This case is illustrated in Figure 3.

Case 3. For any $(\tau, \lambda, \mu) \in T_{n,m,k}$, we add $2k + 1$ to τ to get $\tau' = ((m - k)^2 + (k + 1)^2 + m - k)$. Then we remove a part equals 2 from μ and getting μ' , where $\ell(\mu') = n - m - 1$. Similarly, by removing a part equals to $2k - 1$ from λ , we can get λ' , where $\ell(\lambda') \leq 2k - 1$ and $\lambda'_1 \leq m - 2k$. This leads to the bijection $\varphi_3 : T_{n,m,k} \rightarrow K_{n,m+1,k+1}$ as given by $(\tau, \lambda, \mu) \mapsto (\tau', \lambda', \mu')$. Figure 4 gives an illustration of the bijection φ_3 .

The proof is completed by combining all the above bijections. \blacksquare

Observe that the bijection φ_1, φ_2 and φ_3 preserve the weight. The above theorem immediately leads to a recurrence relation, which implies q -series identity (1.7). To be more specific, we have the following corollary.

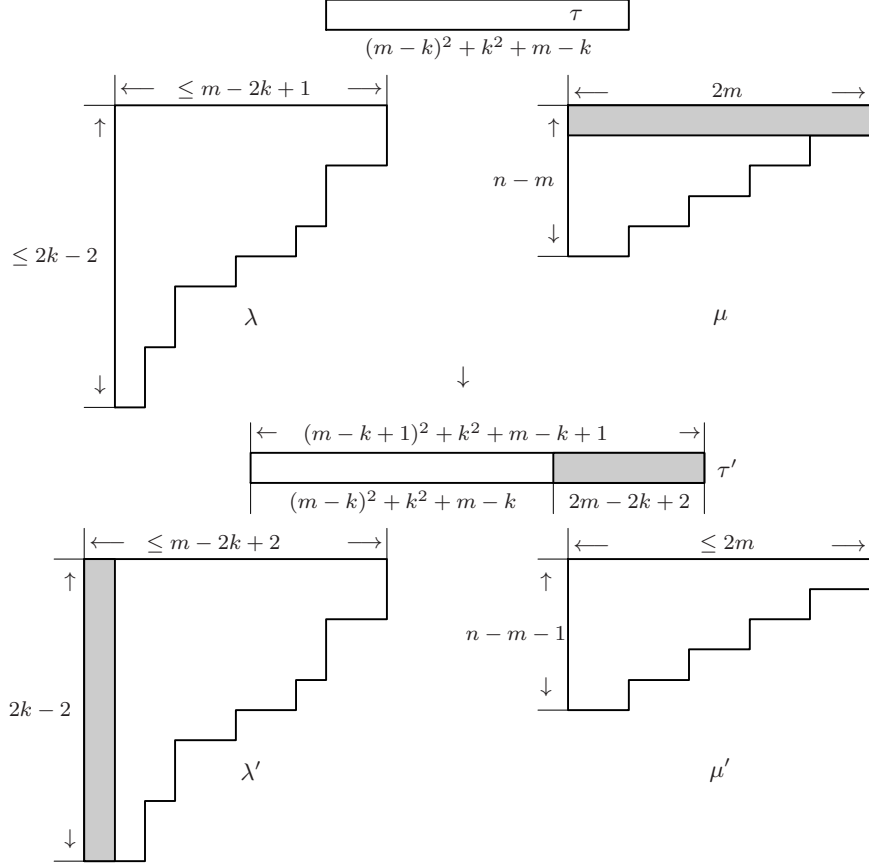


Figure 3: The bijection φ_2 in Case 2.

Corollary 2.2 *Let*

$$F_n(a, q) = \sum_{m, k \geq 0} (-1)^m \sum_{(\tau, \lambda, \mu) \in P_{n, m, k}} a^n q^{|\tau| + |\lambda| + |\mu|},$$

Then for any positive integer n , we have

$$F_n(a, q) = -aq^{2n-1}F_{n-1}(a, q), \quad n \geq 2.$$

Since $F_1(a, q) = -aq$, by iteration we find that

$$F_n(a, q) = (-a)^n q^{n^2}.$$

Summing over n , we arrive at identity (1.7) of Andrews.

For another identity (1.8) of Andrews, we give a sketch of the proof. Set

$$G(q) = \sum_{m, k \geq 0} \frac{(-a)^m q^{(m-k)^2 + k^2 + k}}{(aq^2; q^2)_m} \left[\begin{matrix} m \\ 2k \end{matrix} \right]_q = \sum_{n \geq 0} (-a)^n q^{n^2},$$

and the corresponding summand

$$G_{m, k} = \frac{(-a)^m q^{(m-k)^2 + k^2 + k}}{(aq^2; q^2)_m} \left[\begin{matrix} m \\ 2k \end{matrix} \right]_q = \sum_{n \geq 0} (-a)^n q^{n^2}. \quad (2.2)$$

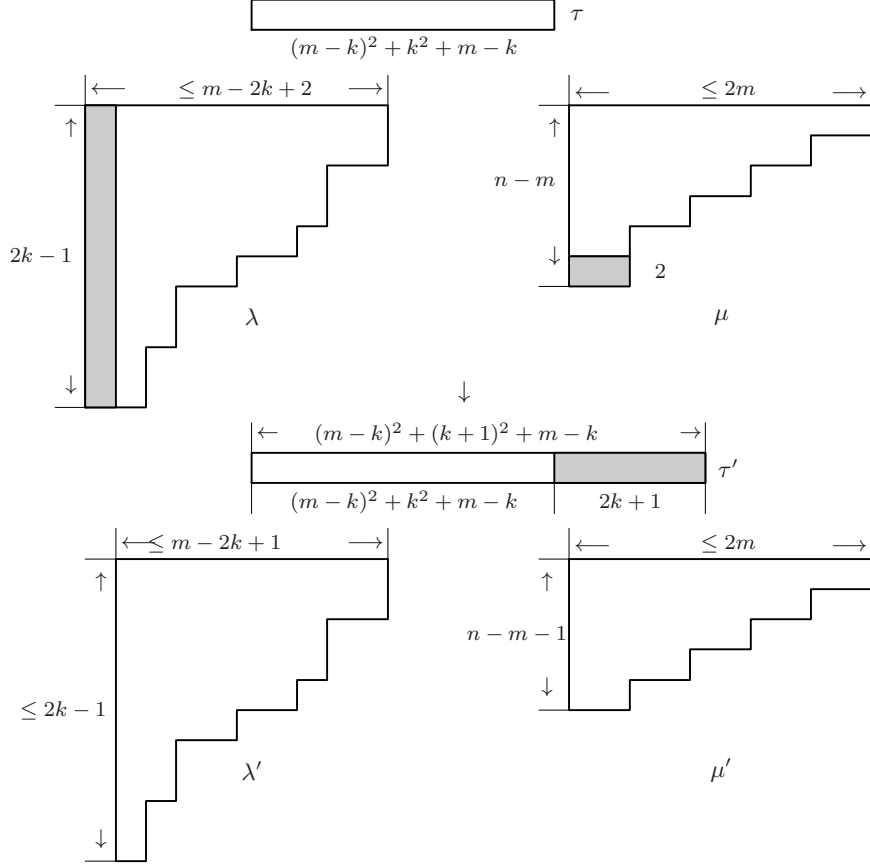


Figure 4: The bijection φ_3 in Case 3.

We also give a combinatorial interpretation of the summand $G_{m,k}$. Let $Q_{m,k}$ be a set of triple (τ, λ, μ) , where τ is a partition with only one part equals $(m-k)^2 + k^2 + k$, λ is a partition with no more than $2k$ parts and each part not exceeding $m-2k$, and μ is a partition with only even parts not exceeding $2m$. Figure 5 gives an illustration of an element of $Q_{m,k}$. One can see that $G_{m,k}$ can be viewed as the weight of $Q_{m,k}$.

Dividing $Q_{m,k}$ into a disjoint union of subsets

$$Q_{n,m,k} = \{(\tau, \lambda, \mu) \in Q_{m,k} : \ell(\mu) = n-m\},$$

where $Q_{n,m,k} = \emptyset$ when $m = k = 0$ or $2k > m > n$. And let

$$M_{n,m,k} = \{(\tau, \lambda, \mu) \in Q_{n,m,k} : \ell(\lambda) = 2k, m_2(\mu) = 0\},$$

$$S_{n,m,k} = \{(\tau, \lambda, \mu) \in Q_{n,m,k} : \ell(\lambda) = 2k-1, m_{2m}(\mu) = 0\},$$

$$L_{n,m,k} = \{(\tau, \lambda, \mu) \in Q_{n,m,k} : \ell(\lambda) \leq 2k-2, m_{2m}(\mu) = 0\}.$$

We have the following combinatorial telescoping relation for $Q_{n,m,k}$.

Theorem 2.3 *For any positive integer n and nonnegative integers m and k ,*

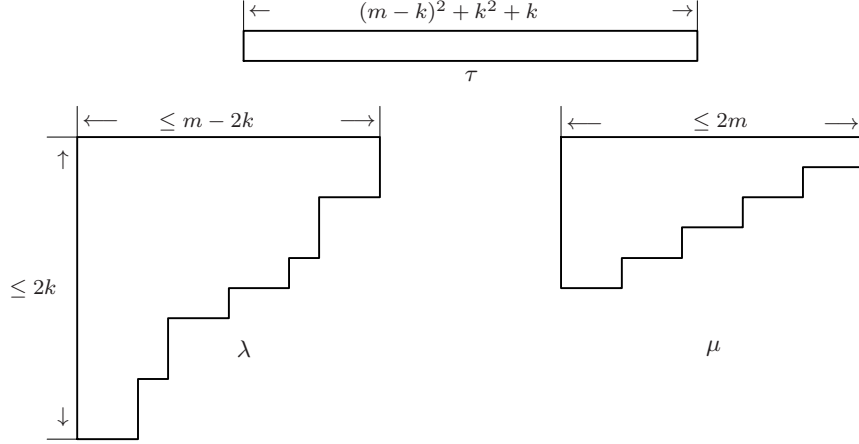


Figure 5: Illustration of an element $(\tau, \lambda, \mu) \in Q_{m,k}$.

there is a bijection

$\psi_{n,m,k} :$

$$Q_{n,m,k} \cup \{2n-1\} \times Q_{n-1,m,k} \cup L_{n,m+1,k} \rightarrow$$

$$M_{n,m,k} \cup M_{n,m+1,k} \cup S_{n,m,k} \cup S_{n,m+1,k} \cup L_{n,m,k} \cup L_{n,m+1,k+1} \cup L_{n,m+1,k}.$$

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. \blacksquare

The above theorem immediately leads to a recurrence relation as follows.

Corollary 2.4 *Let*

$$G_n(a, q) = \sum_{m,k \geq 0} (-1)^m \sum_{(\tau, \lambda, \mu) \in Q_{n,m,k}} a^n q^{|\tau| + |\lambda| + |\mu|}.$$

Then for any positive integer n , we have

$$G_n(a, q) = -aq^{2n-1}G_{n-1}(a, q), \quad n \geq 2.$$

Since $G_1(a, q) = -aq$ and $G_0(a, q) = 1$, by iteration we find that

$$G_n(a, q) = (-a)^n q^{n^2}.$$

Summing over n , we arrive at identity (1.8) of Andrews.

Acknowledgments. We wish to thank Professor William Y.C. Chen for helpful comments and discussions.

References

- [1] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1998.
- [2] G.E. Andrews, Parity in partition identities. Ramanujan J. **23** (2010) 45–90.

- [3] W.Y.C. Chen, Q.-H. Hou and L.H. Sun, The method of combinatorial telescoping, *J. Combin. Theory Ser. A* **118** (2011) 899–907.
- [4] W. Chu, Two problems of George Andrews on generating functions for partitions, *Miskolc Math. Notes*. **13** (2012) 293–302.
- [5] D. Feldman, J. Propp, Producing new bijections from old, *Adv. Math.* **113** (1995) 1–44.
- [6] W.Y.C. Chen, D.K. Du and C.B. Mei, Combinatorial telescoping for an identity of Andrews on parity in partitions, *European J. Combin.* **33** (2012) 510–518.
- [7] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd Edition, Addison-Wesley, Reading, MA, 1994.
- [8] M. Petkovšek, H.S. Wilf, and D. Zeilberger, *A=B*, A.K. Peters, Wellesley, MA, 1996.
- [9] J.J. Sylvester, A constructive theory of partitions, arranged in three acts, an interact, and an exodion, *Amer. J. Math.* **5** (1882) 251–330.
- [10] G.N. Watson, A new proof of the Rogers-Ramanujan identities, *J. London Math. Soc.* **4** (1929) 4–9.
- [11] A.J. Yee, Ramanujans partial theta series and parity in partitions, *Ramanujan J.* **23** (2010) 215–225.
- [12] D. Zeilberger, The method of creative telescoping, *J. Symbolic Comput.* **11**(1991) 195–204.